Applied Mechanics and Materials
ISSN: 1662-7482, Vol. 865, pp 442-449
doi:10.4028/www.scientific.net/AMM.865.442

Submitted: 2016-11-29
Revised: 2017-02-09
Accepted: 2017-02-22
Online: 2017-06-02

Neuron Net for Forming Optimal Smooth Trajectories Based on Bezier Splines

Yukhimets Dmitry[1, a*]

[1]Institute of Automation and Control Processes Far Eastern Branch of RAS, Radio str., 5, Vladivostok, Russia

[a]undim@iacp.dvo.ru

Keywords: Neuron net, Trajectory planning, Optimisation, Bezier splines, Mechatronic object, High speed movement.

Abstract. In this paper, the problem of planning smooth trajectories of mechatronic objects on the basis of Bezier splines of the third order with a minimal curvature is solved. Using such trajectories provides the maximal speed of movement for mechatronic objects. The neuron net, which approximates the function of the optimal selection of spline parameters is proposed to solve this problem. The advantage of the proposed approach is its lack of computational complexity, which allows its use in most on-board computers in real time.

Introduction

Planning trajectories for different mechatronic objects (MO), such as multilink manipulators and automatic transport systems, in an unknown working space is necessary for manufacturing automation. There are several approaches to solving this task. The first approach is to plan the path of the MO as a sequence of waypoints [1, 2]. The advantage of this approach is the simplicity of setting the MO movement. However, a significant disadvantage is the impossibility of predicting real MO trajectory movement. This disadvantage is very strong if the MO moves near different obstacles.

Another approach is to form the MO path as straight parts connected in a sequence of waypoints [3, 4]. However, in this case, the presence of a bend in this path leads to the increase of its acceleration that can negatively influence the quality of operation performance.

The third approach is to form smooth trajectories of MO movements. Using such trajectories allows the MO to move with relatively quick speed. These smooth trajectories can be based on the arc of a circle and straight parts [5] or special functional dependencies [6]. However, the most convenient approach for forming smooth trajectories for any sequence of waypoints is using Bezier splines of the third order. The examples of using the Bezier splines for planning trajectories of different MOs are described in [7, 8]. However, these smooth trajectories are formed without considering the dynamic properties of the MO that can lead to the decrease of its speed of movement or to the increase of dynamic error.

The main negative factor that influences the accuracy and speed of MO movement is the curvature of trajectory. The more curvature, the more magnitude of control signals of MO actuators. This can lead to saturation and deviation of the MO of some actuators from the desired trajectory [9]. Thus, the trajectory of the MO must have the minimal curvature for providing the maximal speed of MO movement. For forming such trajectories, the approaches of numerical optimisation based on gradient-based algorithms [10] are used. However, in case of the formation of the MO trajectory online, these methods are hardly suitable because of their high computational complexity.

Thus, in this paper, a method to form smooth trajectories on the basis of Bezier splines of the third order is developed. This method must have little computational complexity and provide minimal curvature of trajectories.

Task Setting

We consider the part of the MO trajectory set by Bezier splines of the third order through waypoints W_1 and W_2. This spline is described in parametrical form by the following expression [11]:

$$X^*(\tau) = (1-\tau)^3 W_1 + 3(1-\tau)^2 \tau \hat{W}_1 + (1-\tau)\tau^2 \hat{W}_2 + \tau^3 W_2, \tag{1}$$

where, $W_i = [W_{xi}, W_{yi}, W_{zi}]^T$ are coordinates of the base point of the spline; $\hat{W}_i = [\hat{W}_{xi}, \hat{W}_{yi}, \hat{W}_{zi}]^T$, $i = \overline{1,2}$ are coordinates of the control points of the splines, and $X^* = [x, y, z]^T$ are coordinates of the target point moving along the spline, while $\tau \in [0, 1]$ is the variable that defines the movement of the target point X^*.

We define the position of points $\hat{W}_1$ and $\hat{W}_2$, which are described by the following expression:

$$\hat{W}_1 = W_1 + k_{p1}\mathbf{k}_{c1}, \hat{W}_2 = W_2 - k_{p2}\mathbf{k}_{c2}, \tag{2}$$

where, $\mathbf{k}_{ci} = [k_{cxi}, k_{cyi}, k_{czi}]^T$, $\| \mathbf{k}_{ci} \| = 1$, $i = \overline{1,2}$ is the vector of coefficients that defines the direction of MO movement in the initial and end points of the spline, and k_{p1}, k_{p2} is the distance between the corresponding base points and control points of the spline. During the formation of the MO trajectory, the base points $W_i, i = \overline{1,2}$ and vectors $\mathbf{k}_{ci}$ are defined by the mission or data from the on-board sensors of the environment. The curvature of the trajectory can be adjusted using the values k_{p1} and k_{p2}. The examples of splines with equal coordinates of points W_1, W_2 and values of vectors $\mathbf{k}_{cj}$, $j = \overline{1,2}$ but different values of k_{p1} and k_{p2} are shown in Fig. 1.

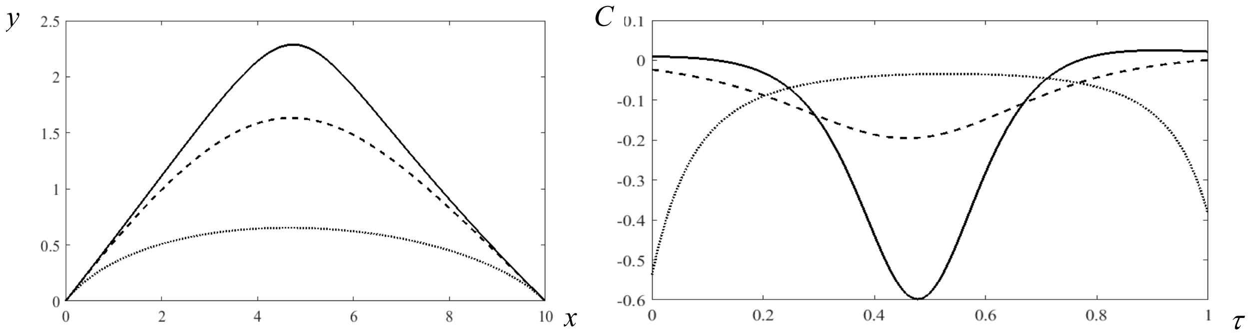


Fig. 1 Form of splines (a) and their curvature C (b) at different values of k_{p1} and k_{p2}: $k_{p1} = k_{p2} = 7$ is a solid line, $k_{p1} = k_{p2} = 5$ is a dashed line, and $k_{p1} = k_{p2} = 0.2$ is a dotted line.

Thus, developing this method of selection of values $k_{pj}, j = \overline{1,2}$ would provide the minimal curvature of the Bezier spline at constant values W_1, W_2, and $\mathbf{k}_{cj}$, $j = \overline{1,2}$. Further, for simplification, we consider the splines located on the horizontal plane xy of the global coordinate frame because such splines are used for planning most trajectories of MO movements. The results can be extended for spatial splines. The curvature C of the trajectory described in the parametric form is calculated using the expression [12]:

$$C(\tau) = \frac{\dot{x}^*(\tau)\ddot{y}^*(\tau) - \dot{y}^*(\tau)\ddot{x}^*(\tau)}{(\dot{x}^*(\tau)^2 + \dot{y}^*(\tau)^2)^{3/2}}, \tag{3}$$

where, $\dot{x}^*(\tau) = \frac{dx^*(\tau)}{d\tau}$, $\dot{y}^*(\tau) = \frac{dy^*(\tau)}{d\tau}$, $\ddot{x}^*(\tau) = \frac{d^2x^*(\tau)}{d\tau^2}$, $\ddot{y}(\tau) = \frac{d^2y^*(\tau)}{d\tau^2}$.

As we can see from Eq. 3, the value C depends not only on the spline parameters but also on parameter τ, (i.e., it is varied along the spline; see Fig. 1b). Thus, it is necessary to provide the minimisation of the maximal value C for a spline with set parameters. For the solution of this optimisation problem, we define the following cost function:

$$J = (C(\tau_{\max}, k_{p1}, k_{p2}))^2, \tag{4}$$

where, $\tau_{\max}$ is the value of variable τ, where C is maximal on the module value, and solving the task is formulated as follows:

$$[k_{p1}, k_{p2}] = \arg\min(J(C(\tau, \mathbf{k}_p))). \tag{5}$$

As we can see from Eq. 1–4, the Eq. 5 is nonlinear and has no analytical solution. The methods of numerical optimisation have large computational complexity and are difficult to use in real time because it is viable to calculate optimal solutions for some variants of parameters W_1, W_2, and $\mathbf{k}_{cj}, j = \overline{1,2}$ using these results for training the neuron net, which provides an approximation of solutions for Eq. 5 for any variance of initial parameters. In the next part, we discuss the method of creation of such a neuron net.

Formation of Training Set for Neuron Net

The main task arising in the process of neuron net training is the formation of a training set. This set must contain input data and reference output data. The coordinates of base points W_1, W_2, and vectors $\mathbf{k}_{cj}, j = \overline{1,2}$ are input data, and the values of k_{p1} and k_{p2} which provide minimal curvature of the spline, are reference output data. The main task arising during formation of this training set is the calculation of optimal values k_{p1} and k_{p2} for different combinations of input data.

To simplify the training set formation and decrease its dimension, we consider the spline passing though the points with coordinates $\widetilde{W}_1 = (0,0)$ and $\widetilde{W}_2 = (\widetilde{W}_{x2}, 0)$. In this case, Eq. 1 and 2 have the following form:

$$\begin{aligned} x(\tau) &= 3(1-\tau)^2 \tau k_{p1}\widetilde{k}_{cx1} + (1-\tau)\tau^2(\widetilde{W}_{x2} - k_{p2}\widetilde{k}_{cx2}) + \tau^3 \widetilde{W}_{x2}, \\ y(\tau) &= 3(1-\tau)^2 \tau k_{p1}\widetilde{k}_{cy1} - (1-\tau)\tau^2 k_{p2}\widetilde{k}_{cy2}. \end{aligned} \tag{6}$$

It should be noted that any spline in the form of Eq. 1 and 2 can be transformed to form Eq. 6 by the following transformation of coordinates (Fig. 2):

$$\widetilde{W}_2 = \begin{bmatrix} \dfrac{W_{x2} - W_{x1}}{\| W_2 - W_1 \|} & \dfrac{W_{y2} - W_{y1}}{\| W_2 - W_1 \|} \\ \dfrac{W_{y1} - W_{y2}}{\| W_2 - W_1 \|} & \dfrac{W_{x2} - W_{x1}}{\| W_2 - W_1 \|} \end{bmatrix} (W_2 - W_1), \widetilde{\mathbf{k}}_{ci} = \begin{bmatrix} \dfrac{W_{x2} - W_{x1}}{\| W_2 - W_1 \|} & \dfrac{W_{y2} - W_{y1}}{\| W_2 - W_1 \|} \\ \dfrac{W_{y1} - W_{y2}}{\| W_2 - W_1 \|} & \dfrac{W_{x2} - W_{x1}}{\| W_2 - W_1 \|} \end{bmatrix} \mathbf{k}_{ci}, i = \overline{1,2}, \tag{7}$$

where, $\widetilde{W}_1 = (0,0,0)^T$ and $\widetilde{W}_2$, $\widetilde{\mathbf{k}}_{ci}, i = \overline{1,2}$ are coordinates of base points W_1, W_2, and vectors $\mathbf{k}_{cj}, j = \overline{1,2}$ in the coordinate frame $\widetilde{X}\widetilde{Y}$ (Fig. 2). In this case, the coordinates of the base point of the spline are defined only by the coordinates of point $\widetilde{W}_2$. Additionally, if values k_{p1} and k_{p2} are presented in the form $k_{p1} = \widetilde{W}_{x2}\widetilde{k}_{p1}$, $k_{p2} = \widetilde{W}_{x2}\widetilde{k}_{p2}$ then Eq. 6 will have the following form:

$$\begin{aligned} \widetilde{x}(\tau) &= 3(1-\tau)^2 \tau \widetilde{k}_{p1} k_{cx1} + 3(1-\tau)\tau^2(1 - \widetilde{k}_{p2} k_{cx2}) + \tau^3, \\ \widetilde{y}(\tau) &= 3(1-\tau)^2 \tau \widetilde{k}_{p1} k_{cy1} - 3(1-\tau)\tau^2 \widetilde{k}_{p2} k_{cy2}. \end{aligned} \tag{8}$$

where, $\widetilde{x}(\tau) = x(\tau) / \widetilde{W}_{x2}$, $\widetilde{y}(\tau) = y(\tau) / \widetilde{W}_{x2}$.

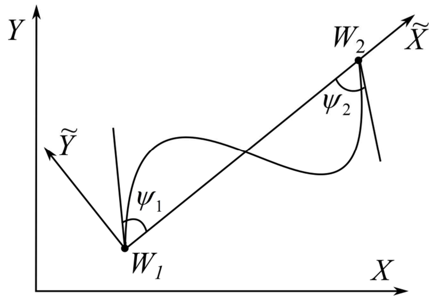


Fig. 2 Transformation of spline coordinates.

As seen, the optimal values $\tilde{k}_{p1}, \tilde{k}_{p2}$ for Eq. 8 do not depend on the value $\widetilde{W}_{x2}$. Therefore, we can exclude value $\widetilde{W}_{x2}$ from the training set. Additionally, we will change the vectors $\tilde{\mathbf{k}}_{ci}, i = \overline{1,2}$ to the values of angles ψ_1, ψ_2 (Fig. 2) of the slope of the tangent lines of the spline for points $\widetilde{W}_1, \widetilde{W}_2$ in the coordinate frame $\widetilde{X}\widetilde{Y}$.

The optimal values of parameters $\tilde{k}_{p1}, \tilde{k}_{p2}$ included in the training set are calculated for each combination of angles ψ_1, ψ_2, whose values are changed in the range [-π/2, π/2] with step 0.04 rad. These optimal values are calculated as follows. For defined combinations of these angles, ψ_1, ψ_2, we find the maximal value $C_{\max}$ of the module of the spline curvature for all combinations of values $\tilde{k}_{p1}, \tilde{k}_{p2}$, which changes in the range [0.2, 1] with step 0.005. Then, we select the combination $\tilde{k}_{p1}, \tilde{k}_{p2}$, which provides the minimal value of $C_{\max}$. This combination is saved in the training set as optimal values of parameters $\tilde{k}_{p1}, \tilde{k}_{p2}$ for the defined combination of angles ψ_1, ψ_2.

Thus, the training set $\mathbf{S}(k_{p1}, k_{p2}, \psi_1, \psi_2)$ for the neuron net was formed. This training set includes input combinations of angles ψ_1, ψ_2 and, corresponding to these combinations, the optimal values $\tilde{k}_{p1}, \tilde{k}_{p2}$, which minimise the spline curvature. The graphical view of the training set for parameter $\tilde{k}_{p1}$ is presented in Fig. 3.

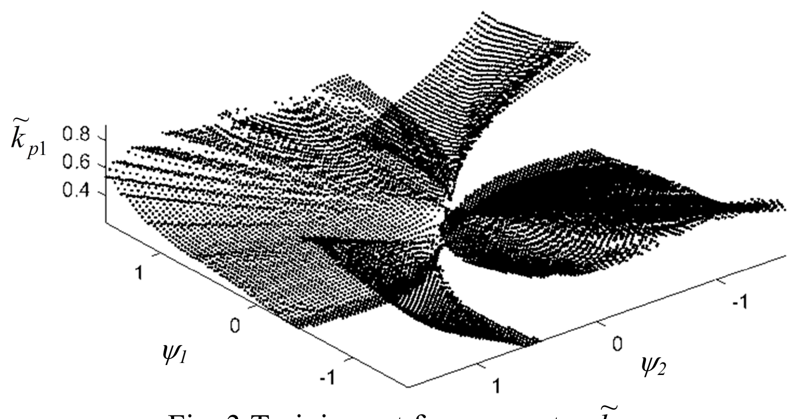


Fig. 3 Training set for parameter $\tilde{k}_{p1}$.

In Fig. 3, the function that describes the dependence of optimal parameters $\tilde{k}_{p1}, \tilde{k}_{p2}$ from the values of the angles ψ_1, ψ_2 has discontinuities that divide the training set into two parts. The presence of these discontinuities shows that optimal parameters $\tilde{k}_{p1}, \tilde{k}_{p2}$ change their values drastically for some values of ψ_1, ψ_2, and this does not provide an accurate approximation of the function of the optimal selection of spline parameters by the neuron net. To solve this problem, it is necessary to divide the training set $\mathbf{S}(k_{p1}, k_{p2}, \psi_1, \psi_2)$ into two parts, $\mathbf{S}_1(k_{p1}, k_{p2}, \psi_1, \psi_2)$ and $\mathbf{S}_2(k_{p1}, k_{p2}, \psi_1, \psi_2)$, which do not include discontinuities. These training sets will be used to synthesise the two neuron nets.

To form training sets $\mathbf{S}_1$ and $\mathbf{S}_2$, it is necessary to obtain the equations of the discontinuity lines for coordinates (ψ_1, ψ_2). Thus, we separate the points that correspond to discontinuities from the training set and approximate these points by analytical function. To define these points, we use the following expressions:

$$D(\psi_1(i),\psi_2(i)) = \begin{cases} 1, if \mid \tilde{k}_{pj}(\psi_1(i),\psi_2(i)) - \tilde{k}_{pj}(\psi_1(i-1),\psi_2(i)) \mid \geq \Delta \\ 1, if \mid \tilde{k}_{pj}(\psi_1(i),\psi_2(i)) - \tilde{k}_{pj}(\psi_1(i),\psi_2(i-1)) \mid \geq \Delta \\ 0, if \mid \tilde{k}_{pj}(\psi_1(i),\psi_2(i)) - \tilde{k}_{pj}(\psi_1(i-1),\psi_2(i)) \mid \leq \Delta \\ 0, if \mid \tilde{k}_{pj}(\psi_1(i),\psi_2(i)) - \tilde{k}_{pj}(\psi_1(i),\psi_2(i-1)) \mid \leq \Delta \end{cases}, \quad j = \overline{1,2}, \tag{9}$$

where, i is the number of elements in the training set, and Δ is the value defining the presence of the drastically changing values $\tilde{k}_{p1}, \tilde{k}_{p2}$.

The result of the separation of the points of discontinuity through Eq. 9 is shown in Fig. 4a. As shown, two discontinuity lines cross at the point (0, 0) and divide the training set into two parts. It is convenient to consider the dataset D(ψ_1, ψ_2) for coordinates ($\tilde{\psi}_1, \tilde{\psi}_2$), which are rotated on the angle $\pi/4$ relative to coordinates (ψ_1, ψ_2) (Fig. 4b).

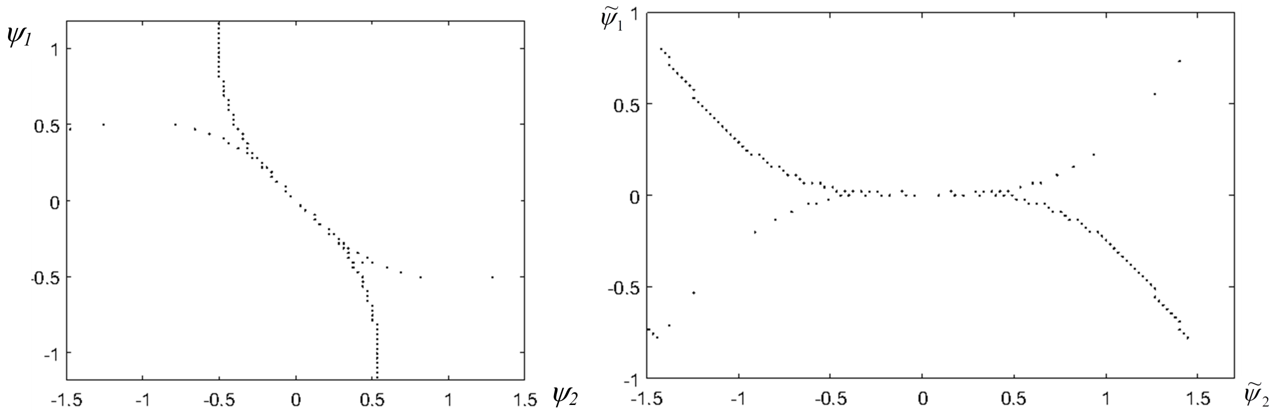


Fig. 4 Discontinuity points in the coordinate system (ψ_1, ψ_2) (a) and $\tilde{\psi}_1, \tilde{\psi}_2$) (b).

For the approximation of discontinuity lines, we select the polynomial function of the third order. The research shows that the use of higher order polynomials leads to increasing the approximation accuracy by no more than 0.5%. Thus, the functions describing the discontinuity lines in coordinates $(\tilde{\psi}_1, \tilde{\psi}_2)$ have following form:

$$\begin{aligned} C_1(\tilde{\psi}_1,\tilde{\psi}_2) &= -\tilde{\psi}_2 + 0.2771\tilde{\psi}_1^{\,3} + 0.00374\tilde{\psi}_1^{\,2} + 0.007866\tilde{\psi}_1, \\ C_2(\tilde{\psi}_1,\tilde{\psi}_2) &= -\tilde{\psi}_2 - (0.2771\tilde{\psi}_1^{\,3} + 0.00374\tilde{\psi}_1^{\,2} + 0.007866\tilde{\psi}_1). \end{aligned} \tag{10}$$

The full training set $\mathbf{S}$ is divided into two parts $\mathbf{S}_1$ and $\mathbf{S}_2$ by the following expressions:

$$\begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{bmatrix} = \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{11}$$

$$\begin{aligned} & S_1(j) = S(i), if\ C_1(\tilde{\psi}_1,\tilde{\psi}_2)C_2(\tilde{\psi}_1,\tilde{\psi}_2) > 0, \\ & S_2(k) = S(i), if\ C_1(\tilde{\psi}_1,\tilde{\psi}_2)C_2(\tilde{\psi}_1,\tilde{\psi}_2) < 0, \\ & i = \overline{1,N},\ j + k = N. \end{aligned} \tag{12}$$

The two training sets are used for training two neuron nets, which approximate the function of the optimal selection of the spline parameters $\tilde{k}_{p1}, \tilde{k}_{p2}$. The description of these nets and the algorithm are shown in the next paragraph.

Synthesis of a Neuron Net for Calculating the Optimal Parameters of the Spline and Algorithm

Because the function of selection of the optimal spline parameters does not depend on time for the approximation, we will use the feedforward neuron net with 20 neurons in the hidden layer (Fig. 5).

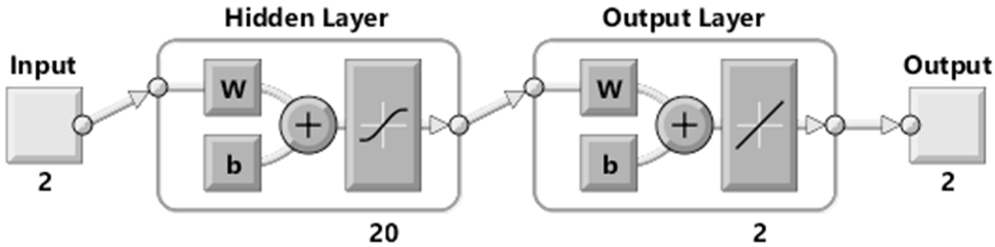
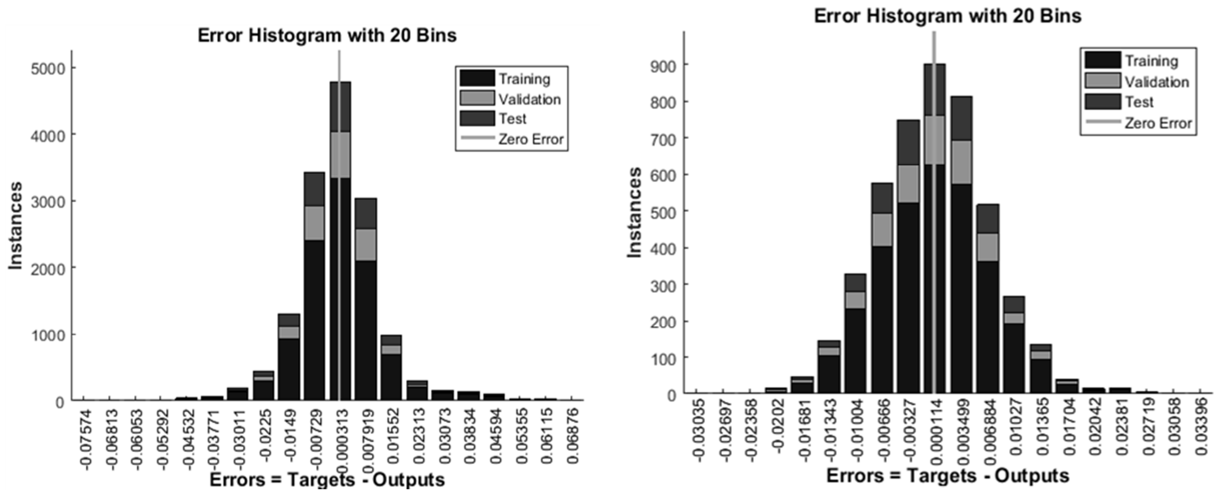


Fig. 5 Structure of a neuron net.

The training error of neuron nets is shown in Fig. 6. As shown, this error does not exceed 0.01 for 90% of the range of the training set. Therefore, the increasing quantity of neurons in the hidden layer in both sets leads to the decrease of the training error to less than 5%.



a) b)

Fig. 6 Training error of neuron nets for training sets $\mathbf{S}_1$ (a) and $\mathbf{S}_2$ (b).

The result of approximation of the training set by synthesised neuron nets is shown in Fig. 7. The algorithm for the calculation of the optimal parameters of the spline using synthesised neuron nets includes the following steps.

Step 1. Normalising the spline parameters using Eq. 7 and 8.

Step 2. Calculating the values of C_1 and C_2 using Eq. 10 and 11.

Step 3. Selecting the neuron net for formation of optimal parameters $\tilde{k}_{p1}, \tilde{k}_{p2}$. The condition of this selection follows: if $C_1C_2 > 0$, then it is necessary to use the neuron net trained on set $\mathbf{S}_1$, and if $C_1C_2 < 0$, then it is necessary to use the neuron net trained on set $\mathbf{S}_2$.

Step 4. The values $\widetilde{k}_{p1}, \widetilde{k}_{p2}$ are multiplied by $\widetilde{W}_{x2}$ and can be used to form the MO trajectory.

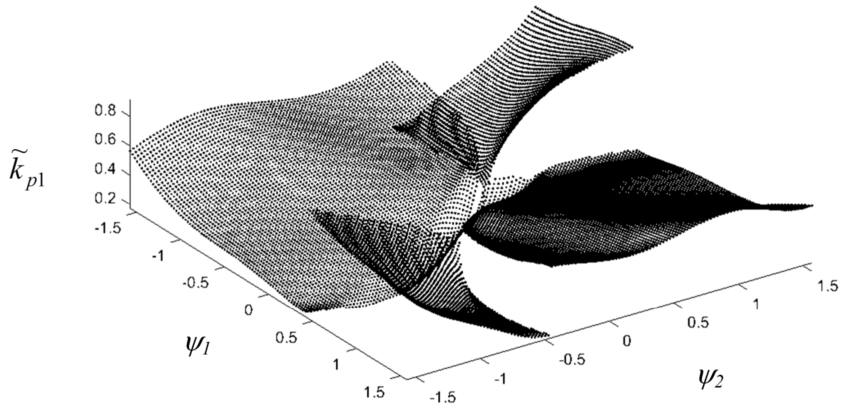


Fig. 7 Approximation using neuron nets for the optimal selection of spline parameters.

Especially, we need to consider the case in which the input of the parameters corresponds to the neighbourhood of discontinuity lines. In this case, we need to use both neuron nets and select the result, which allows the formation of a spline with a smaller curvature.

In the next section, we research the accuracy of the calculation of the optimal spline parameters using the proposed neuron nets.

Research of the Developed Algorithm

The research was carried out to estimate the efficacy of the developed algorithm. The combinations of parameters (ψ_1, ψ_2) that were not present in the training set S were used for the calculation of the parameters $\widetilde{k}_{p1}, \widetilde{k}_{p2}$ during this research. For comparison, the optimal parameters $\widetilde{k}^r_{p1}, \widetilde{k}^r_{p2}$ were calculated using a method that is used for forming a training set. The results of the research are shown in Table 1.

As shown, the use of the proposed algorithm allows the formation of the spline parameters, which are different from the optimal values by not more than 3%. This difference leads to the increase of the maximal curvature of the spline by not more than 0.014 rad/m, which is not significant for the control of most MOs.

Table 1. Results of the investigation of the developed algorithm.

	ψ_1	ψ_2	k_{p1}	k_{p2}	C_{max}	$\widetilde{k}^r_{p1}$	$\widetilde{k}^r_{p2}$	C^r_{max}
1	-1.4	1.4	0.613	0.6081	0.2072	0.61	0.61	0.2063
2	-1.18	0.76	0.2622	0.7594	0.2647	0.27	0.74	0.2658
3	-0.652	0.853	0.2975	0.6747	0.2242	0.3	0.67	0.2243
4	-1.3	-0.2	0.5368	0.2930	0.5260	0.55	0.3	0.5140
5	-1.14	-1.24	0.472	0.5295	0.69	0.47	0.53	0.6875
6	0.36	-1.2	0.2197	0.5596	0.3376	0.2	0.55	0.3356
7	1.11	0.11	0.5044	0.2971	0.4036	0.49	0.3	0.3899
8	0.2	1.31	0.2918	0.5351	0.5308	0.3	0.56	0.52
9	1.46	0.77	0.6476	0.3332	0.7427	0.63	0.33	0.7438
10	0.61	-0.65	0.4175	0.3424	0.1279	0.4	0.35	0.1238
11	0.49	-0.9	0.3503	0.788	0.2546	0.35	0.78	0.2533
12	0.865	-1.46	0.7785	0.3289	0.2628	0.79	0.33	0.2567

Conclusions

The formation of smooth trajectories of MO movements based on the method of the Bezier spline with a minimal curvature is proposed in this paper. The use of such trajectories allows the MO to move with high accuracy and speed. A feature of this method is the use of neuron nets, which allow the avoidance of complex algorithms of numerical optimisation for calculating the spline parameters. Therefore, the proposed algorithm has little computational complexity and can be used in real time on most on-board computers.

Acknowledgements

This work was supported by the Russian Science Foundation (grant No. 14-19-00081).

References

[1] B. Siciliano, O. Khatib, Handbooks of robotics. Springer, 2008.

[2] K. Fu, R. Gonzalez, C. Lee, Robotics: Control, Sensing, Vision, and Intelligence, Mcgraw-Hill Book Company, 1987.

[3] T. I. Fossen, Handbook of Marine Craft Hydrodynamics and Motion Control, Wiey & Sons Ltd., 2011.

[4] I. E. Tuphanov, A. F. Scherbatyuk, Adaptive algorithm of AUV meander pattern trajectory planning for underwater sampling, In Proceedings of the 10th ISOPE Pacific/Asia Offshore Mechanics Symposium, Vladivostok, Russia, (2012), pp. 181 – 185.

[5] A. V. Lebedev, The formation of dynamic objects trajectories in conditions of control signals saturation, In Proc. of the 18th World Multi-Conference on Systemics, Cybernetics and Informatics, Orlando, USA, (1) (2014) 154-159.

[6] V. Munoz, Smooth trajectory planning method for mobile robots, Special issue on Intelligent Autonomous Vehicles of the Journal of Integrated Computed-Aided Engineering, 6(4) (1999) 1-6.

[7] K. Yang, Continuous curvature path-smoothing algorithm using cubic Bézier spiral curves for non-holonomic robots, Adv. Robot. 27(4) (2013) 247-258.

[8] V. Filaretov, D. Yukhimets, E. Mursalimov, A. Scherbatyuk, I. Tuphanov, Some marine trial results of a new method for AUV trajectory motion control, In Proc. of 2014 OCEANS, St. John's, OCEANS 2014. St. John's, NL, Canada, (2014), pp. 1-6.

[9] V. Filaretov, A. Gubankov, I. Gornostaev, The Formation of Motion Laws for Mechatronics Objects Along the Paths with the Desired Speed, In. Proc. of Int. conf. on Computer, Control, Informatics and its Applications, Jacarta, Indonesia, (2016), pp. 93-96.

[10] E. Ikonen, K. Najim, Advances process identification and control, Marcel Dekker Inc., 2002.

[11] C. De Boor, A practical guide to splines, Springer, 2001.

[12] Ph. Korn, M. Korn, Mathematical Handbook for scientists and engineers, MsGrow-Hill Book Co., 1968.